Extending the basic Scott-framework of information systems

Iosif Petrakis
petrakis@mathematik.uni-muenchen.de

June 1, 2011

We study the domain of ideals $|\mathcal{A}|$ of an information system $\mathcal{A}$ from the point of view of a regular and T_1 (therefore, T_2)-extension $\mathcal{N}$ of the standard Scott topology $\mathcal{S}$ on $|\mathcal{A}|$.

$\mathcal{N}$ is defined through the notion of a space in a convergence form, and, as we found out, it is in the general case of a directed complete poset already known as the liminf topology. If $J \in |\mathcal{A}|$, then an $\mathcal{N}$-open set $\mathcal{O}$ can also be characterized by the strong Scott condition

$$J \in \mathcal{O} \Leftrightarrow (\exists \lambda_0 \in \Lambda(J))(\forall \lambda \geq \lambda_0) J_\lambda \in \mathcal{O}$$

where $\Lambda(J)$ is an index set for the directed set of the compact ideals contained in J, ordered by $\lambda \leq \lambda^{'} \Leftrightarrow J_\lambda \subseteq J_{\lambda^{'}}$, for each $\lambda, \lambda^{'} \in \Lambda(J)$. Then $(J_\lambda)_{\lambda \in \Lambda(J)}$ is a topological net, that we call the full net of J.

The study of $\mathcal{N}$ is also natural from an "approximation" point of view. Using the notion of an approximation structure of V. Stoltenberg-Hansen, we show that $(|\mathcal{A}|_0, \mathcal{S})$, where $|\mathcal{A}|_0$ are the compact ideals of $\mathcal{A}$, is an Alexandrov pair of approximations for $|\mathcal{A}|$ (satisfying the Alexandrov and the Scott condition, therefore, the strong Scott condition), while $(|\mathcal{A}|_0, \mathcal{N})$, is a Scott pair of approximations for $|\mathcal{A}|$ (satisfying the strong Scott condition, therefore, the Scott condition, but not the Alexandrov condition).

The aim of our work is to study the degree to which we can extend the basic Scott-framework of $|\mathcal{A}|$, centered around the use of $\mathcal{S}$ on $|\mathcal{A}|$ and on the function space $C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|)$ of $\mathcal{S}$-continuous functions, to a corresponding framework centered around $\mathcal{N}$ instead of $\mathcal{S}$.

Besides $\mathcal{S} \subset \mathcal{N}$, we also prove that $C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|) \subset C_{\mathcal{N}}(|\mathcal{A}|, |\mathcal{B}|)$ and if $f \in C_{\mathcal{N}}(|\mathcal{A}|, |\mathcal{B}|)$, then $f \in C_{\mathcal{S}}(|\mathcal{A}|, |\mathcal{B}|)$ iff f is monotone.

Using a standard result on regular spaces we show that a function $f_0 : |\mathcal{A}|_0 \to |\mathcal{B}|$ satisfying a condition we call $\mathcal{N}$-monotonicity, has a unique $\mathcal{N}$-continuous extension $\widehat{f_0} : |\mathcal{A}| \to |\mathcal{B}|$. Also, each monotone $f_0 : |\mathcal{A}|_0 \to |\mathcal{B}|$ is $\mathcal{N}$-monotone.

The exact structure of $(C_{\mathcal{N}}(|\mathcal{A}|, |\mathcal{B}|), \leq)$, where $\leq$ is the pointwise order, is still under study.